Problem 1 – Graph&Co

The λ Guys – MECL

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We have completely treated questions 1, 2, 3 and 5 except for the reformulation of $3b/c$

In Part 1, we will reformulate the problem in graph theory. In Part 2, treat the questions in relation to the optimal transit time. Then, in Part 3, we have treated questions in relation to the optimal effort time.

Here is the prompt:

1. Graph&Co

Younes is responsible for logistics within the company Graph&Co. The company consists of $N \geq 1$ departments and is about to move to new premises, a tower with N floors. Younes's mission is to plan the move by distributing the N departments among the N floors.

However, Younes wants to make the distribution intelligently. He knows that certain departments communicate frequently and therefore employees must often make trips back and forth between them. We call the *communication plan* the set of pairs of departments that communicate.

Since the tower only contains stairs, to spare the employees, Younes seeks to place departments that communicate on floors as close as possible. Given a communication plan, the *transit time* of a distribution is the largest floor difference among the pairs of departments that communicate. The *optimal transit time* of the plan is the smallest transit time that Younes can obtain by distributing the departments.

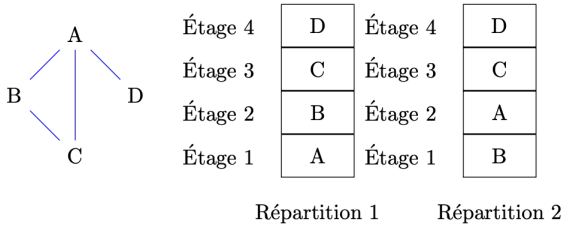

FIGURE 1 – Exemples de répartition

For example, suppose there are four departments A, B, C, D and that the pairs of departments that communicate are $\{A, B\}$, $\{A, C\}$, $\{A, D\}$, $\{B, C\}$. If Younes distributes the departments, from the first to the N-th floor, in the order $A - B - C - D$ then the transit time is 3 (between departments A and D). But if he distributes them in the order $B - A - C - D$ then the transit time is 2 and it is impossible to do better. Consequently, the optimal transit time of this configuration is 2.

0. Briefly explain how to reformulate the problem statement using graph theory terminology. Reformulating certain questions that follow in terms of graph theory can facilitate their treatment. This reformulation is recommended in this case. Known results from graph theory may be admitted in this case.

1. Calculate or bound the optimal transit time as a function of N for each of the following communication plans:

   a) One of the departments communicates with each other department.

   b) By numbering the departments from 1 to N, every department i communicates with department $i+1$ and department N communicates with department 1.

   c) The departments are formed of two groups of size k and l (therefore $N = k + l$) and each department of one group communicates with all the departments of the other group, and does not communicate with any department of its group.

   d) The number of departments is a square $N = n^2$, the departments are labeled with pairs (i, j) where $1 \leq i, j \leq n$ and department (i, j) communicates with departments $(i+1, j)$ (if $i+1 \leq n$) and $(i, j+1)$ (if $j+1 \leq n$).

   e) The number of departments is a cube $N = n^3$, the departments are labeled with triplets (i, j, k) where $1 \leq i, j, k \leq n$ and department (i, j, k) communicates with departments $(i+1, j, k)$ (if $i+1 \leq n$), $(i, j+1, k)$ (if $j+1 \leq n$) and $(i, j, k+1)$ (if $k+1 \leq n$).

2. Describe the plans whose optimal transit time is

   a) 1,

   b) 2,

   c) $N - 2$,

   d) $N - 1$.

3. For each of the following constraints, bound as a function of N and the other variables used the worst optimal transit time (therefore the largest) among the set of communication plans that verify this constraint:

   a) There are at most k pairs of departments that communicate.

   b) Each department communicates with at most k other departments.

   c) There are exactly $N - 1$ pairs of departments that communicate and all departments are connected, in the sense that for any two departments X and Y, one can find departments $D_1, \ldots, D_l$ such that D_i communicates with D_{i+1} for all i.

For a given communication plan, we call the *inverse plan* the plan in which two departments communicate if, and only if, they did not communicate in the initial plan.

4. For which pairs (t, t') can one find a communication plan with optimal transit time t whose inverse plan has an optimal transit time t'?

Concerned about the physical health of his employees, the CEO of Graph&Co wants them to take the stairs as much as possible. He asks Younes on the contrary to ensure that the departments that communicate are as far apart as possible. Given a communication plan, the *effort time* of a distribution is the smallest floor difference among the pairs of departments that communicate. The *optimal effort time* of the plan is the largest effort time that Younes can obtain by distributing the departments.

5. Revisit questions 1., 2., 3. in this framework, replacing the transit time by the effort time.

6. For which pairs (t, t') can one find a communication plan with optimal transit time t and optimal effort time t'?

7. Propose and study other research directions.

Contents

1 Part 1 - Reformulation in graph theory **4**

2 Part 2 - Optimal transit time **4**

3 Part 3 - Optimal Effort Time **17**

1 Part 1 - Reformulation in graph theory

In the context of question **0.** and to treat the problem more efficiently , we will reformulate the statement of the problem using the terminology of graph theory:
We model the departments in communication by the simple graph $G(E, P)$ with:
– $E = \{D_1, D_2, D_3, D_4, \dots, D_N\}$ where E represents the set of departments
– $P = \{C_{D_1,D_2}, C_{D_2,D_3}, C_{D_1,D_3}, C_{D_2,D_4},, \dots\}$ where P represents the communication plan
We model the tower with N floors by the set T defined by:
– $T = \{1, 2, 3, \dots, N\}$ such that: $|T| = |E|$
We distribute the departments to the floors using the function $h(D)$ defined by:
– $(\forall D \in E)(\exists n \in T)/ \quad h(D) = n$
We define the travel time using the function f defined by:
– $f : P \longrightarrow [\![1, n-1]\!]$
$C_{D_i,D_j} \longmapsto |h(D_i) - h(D_j)|$

- **Optimal transit time**
  We find that the transit time of a certain distribution is $T_{trans} = max(f(C_{D_i,D_j}))$
  The transit time is the minimal value of T_{trans} being $T_{min} = min(T_{trans})$
- **Optimal effort time**
  We find that the effort time of a certain distribution is $T_{eff} = min(f(C_{D_i,D_j}))$
  The transit time is the minimal value of T_{eff} being $T_{max} = max(T_{eff})$

□

2 Part 2 - Optimal transit time

1. Calculate or bound the optimal transit time as a function of N for each of the following communication plans :

Before treating this question, let's describe in general the way used to answer it because certain elements of the proofs will be reused later for other questions notably those in relation to the optimal effort time. We will operate in the following way:
– Reformulate the situation mathematically and determine the set of its characteristics
– Prove the result obtained using various methods (mainly by finding the optimal arrangement and proving subsequently by absurdity that it is not possible to do better)

a) One of the departments communicates with each other department.

Answer 2.1. $T_{min} = \lfloor \frac{N}{2} \rfloor$

Proof:

The departments are distributed in such a way that $(\forall a \in T) : h(D_a) = a$. All departments communicate with a single other and do not communicate with each other, so their role is interchangeable.
Let's name the department that communicates with the others $D_i \in T$

$P = \{C_{D_i,D_1}, C_{D_i,D_2}, C_{D_i,D_3}, \ldots, C_{D_i,D_{i-1}}, C_{D_i,D_{i+1}}, \ldots, C_{D_iD_N}\}$

1^{st} case : $i = \lfloor \frac{N}{2} \rfloor$

$(\forall k \in]i; N[) : f(C_{D_iD_k}) < f(C_{D_iD_N})$
And: $f(C_{DiDN}) = |h(D_i) - h_(D_N)|$
$\quad f(C_{DiDN}) = |\lfloor \frac{N}{2} \rfloor - N| \quad f(C_{DiDN}) = N - \lfloor \frac{N}{2} \rfloor$
$(\forall j \in]1; i[) : f(C_{D_iDj}) < f(C_{D_1Di})$
And: $f(C_{D_iD_1}) = |h(D_1) - h(D_i)|$
$\quad f(C_{DiD_N}) = |\lfloor \frac{N}{2} \rfloor - 1| \qquad f(C_{DiD_N}) = |\lfloor \frac{N}{2} \rfloor - 1|$

$(\forall n \in T - i) : T_{\text{transit}_1} = \max(f(C_{D_iD_n}))$
$\qquad = max(N - \lfloor \frac{N}{2} \rfloor, \lfloor \frac{N}{2} \rfloor - 1) = N - \lfloor \frac{N}{2} \rfloor$
$(N - \lfloor \frac{N}{2} \rfloor) - (\lfloor \frac{N}{2} \rfloor - 1) = N - 2\lfloor \frac{N}{2} \rfloor + 1 = 1 > 0 \implies (N - \lfloor \frac{N}{2} \rfloor) > (\lfloor \frac{N}{2} \rfloor - 1)$

2^{nd} case : $i < \lfloor \frac{N}{2} \rfloor$

$(\forall k \in]i; N[) : f(C_{D_iD_k}) < f(C_{D_iD_N})$
And: $f(C_{DiD_N}) = |h(D_i) - h_(D_N)|$
$\quad f(C_{DiD_N}) > |\lfloor \frac{N}{2} \rfloor - N| \quad f(C_{DiD_N}) > N - \lfloor \frac{N}{2} \rfloor$
$(\forall j \in]1; i[) : f(C_{D_iD_j}) < f(C_{D_1D_i})$
And: $f(C_{D_iD_1}) = |h(D_1) - h(Di)|$
$\quad f(C_{DiD_1}) < |\lfloor \frac{N}{2} \rfloor - 1| \qquad f(C_{DiD_1}) < |\lfloor \frac{N}{2} \rfloor - 1|$

$(\forall n \in T - i) : T_{\text{transit}_2} = \max(f(C_{D_iD_n}))$
$\qquad = max(f(C_{DiD_1}), f(C_{DiD_N})) > N - \lfloor \frac{N}{2} \rfloor$

3^{rd} case : $i > \lfloor \frac{N}{2} \rfloor$

$(\forall k \in]i; N[) : f(C_{D_iD_k}) < f(C_{D_iD_N})$
And: $f(C_{DiD_N}) = |h(D_i) - h_(D_N)|$
$\quad f(C_{DiD_N}) < |\lfloor \frac{N}{2} \rfloor - N| \quad f(C_{DiD_N}) < N - \lfloor \frac{N}{2} \rfloor \quad f(C_{DiD_N}) < \lfloor \frac{N}{2} \rfloor$
$(\forall j \in]1; i[) : f(C_{D_iDj}) < f(C_{D_1Di})$
And: $f(C_{D_iD_1}) = |h(D1) - h(Di)|$
$\quad f(C_{DiD_1}) > |\lfloor \frac{N}{2} \rfloor - 1| \quad f(C_{DiD_1}) > \lfloor \frac{N}{2} \rfloor - 1 \qquad f(C_{DiD_1}) \geq \lfloor \frac{N}{2} \rfloor$

$(\forall n \in T - i) : T_{\text{transit}_2} = \max(f(C_{D_iD_n}))$
$\qquad = max(f(C_{DiD_1}), f(C_{DiD_N})) \geq \lfloor \frac{N}{2} \rfloor$

By disjunction of cases, we conclude that: $T_{min} = min(T_{\text{transit}_1}, T_{\text{transit}_2}, T_{\text{transit}_3})$
From which : $T_{min} = N - \lfloor \frac{N}{2} \rfloor = \lfloor \frac{N}{2} \rfloor$

□

b) By numbering the departments from 1 to N, every department i communicates with department $i + 1$ and department N communicates with department 1.

Answer 2.2. $T_{min} = 2$

Proof:

In this case, P is written in the form:
$P = \{C_{D_1,D_2}, C_{D_2,D_3}, \ldots, C_{D_i,D_{i+1}}, C_{D_{N-1},D_N}, C_{D_1,D_N}\}$ where $1 \leq i \leq N-1$

Let's show that $T_{min} \leq 2$:

We consider the following distribution: $\begin{cases} h(D_i) = 2i - 1; & 0 < k \leq \lceil \frac{N}{2} \rceil \\ h(D_i) = 2(N - i - 1); & \lceil \frac{N}{2} \rceil + 1 \leq k \leq N \end{cases}$

$$(\forall i \in [\![1, \lceil \frac{N}{2} \rceil]\!]); \quad f(C_{D_i,D_{i+1}}) = |h(D_i) - h(D_{i+1})|$$

$$\implies (\forall i \in [\![1, \lceil \frac{N}{2} \rceil]\!]); \quad f(C_{D_i,D_{i+1}}) = |h(D_i) - h(D_{i+1}) = |2i - 1 - 2(i+1) + 1|$$

$$\implies (\forall i \in [\![1, \lceil \frac{N}{2} \rceil]\!]); \quad f(C_{D_i,D_{i+1}}) = 2 \tag{1}$$

And:

$$(\forall i \in [\![\lceil \frac{N}{2} \rceil + 1, N]\!]); \quad f(C_{D_i,D_{i+1}}) = |h(D_i) - h(D_{i+1})|$$

$$\implies (\forall i \in [\![\lceil \frac{N}{2} \rceil + 1, N]\!]); \quad f(C_{D_i,D_{i+1}}) = |2(n - i + 1) - 2(n - (i+1) + 1)|$$

$$\implies (\forall i \in [\![\lceil \frac{N}{2} \rceil + 1, N]\!]); \quad f(C_{D_i,D_{i+1}}) = 2 \tag{2}$$

And: $\quad 1 \leq \lceil \frac{N}{2} \rceil \leq \lceil \frac{N}{2} \rceil$ and $\lceil \frac{N}{2} \rceil + 1 \leq \lceil \frac{N}{2} \rceil + 1 \leq N$

$$\implies f(C_{D_{\lceil \frac{N}{2} \rceil}, D_{\lceil \frac{N}{2} \rceil, \lceil \frac{N}{2} \rceil + 1}}) = |2\lceil \frac{N}{2} \rceil| - 1 - 2(n - \lceil \frac{N}{2} \rceil - 1 + 1|$$

$$\implies f(C_{D_{\lceil \frac{N}{2} \rceil}, D_{\lceil \frac{N}{2} \rceil, \lceil \frac{N}{2} \rceil + 1}}) = 1 \tag{3}$$

And: $\quad 1 \leq 1 \leq \lceil \frac{N}{2} \rceil$ and $\lceil \frac{N}{2} \rceil + 1 \leq N \leq N$

$$\implies f(C_{D_1,D_N}) = |2 - 1 - 2(n - n + 1)|$$

$$\implies f(C_{D_1,D_N}) = 1 \tag{4}$$

From the results obtained, we conclude that: $T_{trans} = 2$
From which: $\quad T_{min} \leq 2$

Let's show by absurdity that $T_{min} \geq 2$:

We suppose that $T_{min} \leq 1$
Therefore: $T_{min} = 1$
$T_{min} = 1 \implies (\forall C_{D_i,D_j} \in P); f(C_{D_i,D_j}) = 1$
$\implies (\forall C_{D_i,D_j} \in P); |h(D_i) - h(D_j)| = 1$
$\implies |h(D_1) - h(D_N)| = 1$ and $|h(D_i) - h(D_{i+1})| = 1$ with $1 \leq i \leq N-1$
We will only treat the case where $h(D_N) - h(D_1) = 1$ and $h(D_{i+1}) - h(D_i) = 1$ because the

others are deduced by analogy:
There necessarily exists a D_k/ $h(D_k) = N$
– If $1 \leq k \leq N-1$
$h(D_{k+1}) - h(D_k) = 1$
$\implies h(D_{k+1}) = N+1$ which is absurd
– If $k = N$
$h(D_N) = N$
$\implies h(D_1) = N-1$
and $h(D_{N-1}) = N-1 \implies h(D_1) = h(D_{N-1})$ which is absurd
Conclusion: $T_{min} \geq 2$

$T_{min} \leq 2$ and $T_{min} \geq 2 \implies T_{min} = 2$ □

c) The departments are formed of two groups of size k and l (therefore $N = k + l$) and each department of one group communicates with all the departments of the other group, and does not communicate with any department of its group.

Answer 2.3. $T_{min} = min(\lceil \frac{l}{2} + k - 1 \rceil, \lceil \frac{k}{2} + l - 1 \rceil)$

Proof:

We can write the two groups k and l in the form of two subsets:
$E_k = \{D_{k,i} \quad / i \in \{1, 2, \ldots, K\}\}$
$E_l = \{D_{l,j} \quad / l \in \{1, 2, \ldots, l\}\}$
With: $E = E_k \cup E_l$ set of all departments.
The communication plan is written in this case in the form:
$P = \{C_{D_{k,i}, D_{l,j}} \quad / D_{k,i} \in E_k and D_{l,j} \in E_l\}$
The travel time is written:
$f(C_{D_{k,i}, D_{l,j}}) = |h(D_{k,i}) - h(D_{l,j})|$

We notice that the roles of k and l are interchangeable, therefore to arrive at the result obtained, we will prove only one of the two results being $T_{min} = \lceil \frac{l}{2} + k - 1 \rceil$ according to a condition that we will specify later.

Let's show that $T_{min} \leq \lceil \frac{l}{2} + k - 1 \rceil$

We will study the distribution where the group of smaller size is found in the middle.
The roles of k and l are interchangeable. It is therefore sufficient to study the optimal transit time T_{min} in the case where $k < l$
Let's order the departments of k and l so that:
– $h(D_{k,i})$ strictly increasing bounded below by $h(D_{k,1})$ and bounded above by $h(D_{k,k})$
– $h(D_{l,i})$ strictly increasing bounded below by $h(D_{l,1})$ and bounded above by $h(D_{l,l})$
$(\forall j \leq \lfloor \frac{l}{2} \rfloor)$; $\quad h(D_{l,j}) \leq h(D_{l,i})$
$(\forall j \geq \lceil \frac{l}{2} \rceil)$; $\quad h(D_{l,j}) \geq h(D_{l,i})$ (these two roles are interchangeable)
Let's take $j' \in \{1, 2, \ldots, \lfloor \frac{l}{2} \rfloor\}$:
$f(C_{D_{k,i}, D_{l,j'}}) = |h(D_{k,i}) - h(D_{l,j'})|$
$\implies f(C_{D_{k,i}, D_{l,j'}}) = h(D_{k,i}) - h(D_{l,j'})$
$\implies max(f(C_{D_{k,i}, D_{l,j'}})) = h(D_{k,k}) - h(D_{l,1})$

$\implies max(f(C_{D_{k,i},D_{l,j'}})) = \lfloor \frac{l}{2} + k - 1 \rfloor$
In the same way, we find $max(f(C_{D_{k,i},D_{l,j"}})) = \lceil \frac{l}{2} + k - 1 \rceil$ by taking $j" \in \{\lceil \frac{l}{2} + 1 \rceil, \lceil \frac{l}{2} + 2 \rceil, \ldots, l\}$
We therefore conclude that $T_{min} \leq \lceil \frac{l}{2} + k - 1 \rceil$

Let's show by absurdity that $T_{min} \geq \lceil \frac{l}{2} + k - 1 \rceil$:
Let's suppose that $T_{min} \leq \lceil \frac{l}{2} + k - 2 \rceil$ For a given T_{min}, there necessarily exists a $D_{k,a}$ and a $D_{l,b}$ such that:

$$|h(D_{k,a}) - h(D_{l,b})| = T_{min}$$

We will restrict our study to the case where $h(D_{l,b}) > h(D_{k,a})$ because the other case is symmetric.
Then: $h(D_{l,b}) - h(D_{k,a}) = T_{min}$
Let's suppose that: $(\exists D_c, D_d \in E) \ / \ h(D_d) \geq h(D_{l,b}) + 1$ and $h(D_c) \leq h(D_{k,a}) - 1$

$$h(D_d) - h(D_{k,a}) \geq h(D_{l,b}) + 1 - h(D_{k,a})$$
$$\implies h(D_d) - h(D_{k,a}) \geq T_{min} + 1$$
$$\implies D_d \in E_k$$

(because they must not communicate)

Similarly:

$$h(D_{l,b}) - h(D_c) \geq h(D_{l,b}) - h(D_{k,a}) + 1$$
$$\implies h(D_{l,b}) - h(D_{k,a}) \geq T_{min} + 1$$
$$\implies D_c \in E_l$$

(because they must not communicate)

However: $D_c \in E_l$ and $D_d \in E_k \implies D_c$ and D_d communicate.
$\implies (\exists C_{D_c,D_d} \in P) \ / \ f(C_{D_c,D_d}) = |h(D_c) - h(D_d)|$
$\implies (\exists C_{D_c,D_d} \in P) \ / \ f(C_{D_c,D_d}) = h(D_d) - h(D_c)$
$\implies (\exists C_{D_c,D_d} \in P) \ / \ f(C_{D_c,D_d}) \geq h(D_{l,b}) + 1 - h(D_{k,a}) + 1$
$\implies (\exists C_{D_c,D_d} \in P) \ / \ f(C_{D_c,D_d}) \geq T_{min} + 2$ which is absurd
In this case, for an optimal transit time, it is necessary that one of the two elements occupies an extremity. We will again study only the case where $h(D_{k,a}) = 1$ because the other is symmetric.
$h(D_{k,a}) = 1$
– If $k < N - h(D_{l,b})$
Then: $(\exists D_{l,e}) \ / \ h(D_{l,e}) > h(D_{l,b}) \implies h(D_{l,e}) - h(D_{k,a}) > T_{min}$ which is absurd
– If $k > N - h(D_{l,b})$
Then: $(\exists D_{k,f} \in E_l) \ / \ h(D_{k,f}) = N$

$$(\forall D_{l,j} \in E_l); h(D_{k,f}) - h(D_{l,j}) \leq \lceil \frac{l}{2} + k - 2 \rceil$$
$$\implies (\forall D_{l,j} \in E_l); N - h(D_{l,j}) \leq \lceil \frac{l}{2} + k - 2 \rceil$$
$$\implies (\forall D_{l,j} \in E_l); l + k - h(D_{l,j}) \leq \lceil \frac{l}{2} + k - 2 \rceil$$
$$\implies (\forall D_{l,j} \in E_l); h(D_{l,j}) \geq \lceil \frac{l}{2} + 2 \rceil$$

$$\implies (\forall D_{l,j} \in E_l); \lceil \frac{l}{2} + 2 \rceil \leq h(D_{l,j}) \leq \lceil \frac{l}{2} + k \rceil$$

This implies that at most: $l = \lceil \frac{l}{2} + k \rceil - \lceil \frac{l}{2} + 2 \rceil + 1$
From which: $l = k - 2$ which is absurd because we assumed that $k > l$
Conclusion: $T_{min} \geq \lceil \frac{l}{2} + k - 1 \rceil$

$T_{min} \leq \lceil \frac{l}{2} + k - 1 \rceil$ and $T_{min} \geq \lceil \frac{l}{2} + k - 1 \rceil \implies T_{min} = \lceil \frac{l}{2} + k - 1 \rceil$

This is true for $l > k$. By analogy, for $l < k$: $T_{min} = \lceil \frac{k}{2} + l - 1 \rceil$
Let's try to find a general form of T_{min}:
$l > k \implies \lceil \frac{k}{2} + l - 1 \rceil > \lceil \frac{l}{2} + k - 1 \rceil$ and $T_{min} = \lceil \frac{l}{2} + k - 1 \rceil$
Therefore $T_{min} = min(\lceil \frac{l}{2} + k - 1 \rceil, \lceil \frac{k}{2} + l - 1 \rceil)$ in this case.
$l < k \implies \lceil \frac{k}{2} + l - 1 \rceil < \lceil \frac{l}{2} + k - 1 \rceil$ and $T_{min} = \lceil \frac{l}{2} + k - 1 \rceil$
Therefore $T_{min} = min(\lceil \frac{l}{2} + k - 1 \rceil, \lceil \frac{k}{2} + l - 1 \rceil)$ in this case.
We can therefore conclude that: $T_{min} = min(\lceil \frac{l}{2} + k - 1 \rceil, \lceil \frac{k}{2} + l - 1 \rceil)$

d) The number of departments is a square $N = n^2$, the departments are labeled with pairs (i, j) where $1 \leq i, j \leq n$ and department (i, j) communicates with departments $(i+1, j)$ (if $i+1 \leq n$) and $(i, j+1)$ (if $j+1 \leq n$).

Answer 2.4. $T_{min} = n$

Proof:

In this case, the set of departments and the communication plan are written in the form:
– $E = \{D_{i,j} / 1 \leq i, j \leq n\}$
– $D = P_x \cup P_y$
With $P_x = \{C_{D_{i,j},D_{i+1,j}} \ / \ i + 1 \leq n\}$
$P_y = \{C_{D_{i,j},D_{i,j+1}} \ / \ j + 1 \leq n\}$

Let's show that $T_{min} \leq n$:
We consider the following distribution:
$(\forall D_{i,j} \in E), h(D_{i,j}) = (i-1)n \ + \ j$
$(\forall C_{D_{i,j},D_{i+1,j}} \in P_x); f(C_{D_{i,j},D_{i+1,j}}) = |h(D_{i,j}) - h(D_{i+1,j})|$
$\implies (\forall C_{D_{i,j},D_{i+1,j}} \in P_x); f(C_{D_{i,j},D_{i+1,j}}) = |(i-1)n \ + \ j \ - \ (i-1+1)n \ - \ j|$
$\implies (\forall C_{D_{i,j},D_{i+1,j}} \in P_x); f(C_{D_{i,j},D_{i+1,j}}) = |-n|$
$\implies (\forall C_{D_{i,j},D_{i+1,j}} \in P_x); f(C_{D_{i,j},D_{i+1,j}}) = n$

$(\forall C_{D_{i,j},D_{i,j+1}} \in P_y); f(C_{D_{i,j},D_{i,j+1}}) = |h(D_{i,j}) - h(D_{i,j+1})|$
$\implies (\forall C_{D_{i,j},D_{i+1,j}} \in P_x); f(C_{D_{i,j},D_{i+1,j}}) = |(i-1)n \ + \ j \ - \ (i-1)n \ - \ j - \ 1|$
$\implies (\forall C_{D_{i,j},D_{i+1,j}} \in P_x); f(C_{D_{i,j},D_{i+1,j}}) = |-1|$
$\implies (\forall C_{D_{i,j},D_{i+1,j}} \in P_x); f(C_{D_{i,j},D_{i+1,j}}) = 1$

Therefore, from the results obtained: $(\forall C_{D_{i,j}}, C_{D_{i',j'}} \in P); f(C_{D_{i,j}}, C_{D_{i',j'}}) \leq n$
$\implies T_{transit} = n$
From which, whatever the distribution : $T_{min} \leq n$

Let's show that $T_{min} \geq n$
Let's take up the communication plan in an illustrated way: a grid

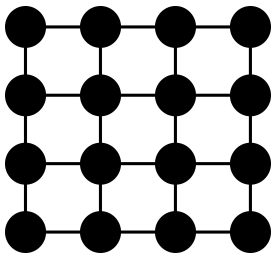

Figure 1: Communication plan for $n = 4$

For the continuation of our proof, we need the following result:
– If we separate our grid into two equal parts (or almost equal if N odd). We will necessarily "break" at least 4 communications. That is to say that the first group shares at least with the second group n communications.
This result can be obtained by applying corollary 4 Bollobàs–Leader [1] which says:

Lemma 2.1. We consider $A \subseteq [k]^n$. Then:

$$|\partial_e(A)| \geq \begin{cases} \frac{4|A|}{k} & \text{if } |A| < \frac{k^n}{4}, \\ k^{n-1} & \text{if } \frac{k^n}{4} \leq |A| < \frac{3k^n}{4}, \\ \frac{4(k^n - |A|)}{k} & \text{if } |A| > \frac{3k^n}{4}. \end{cases}$$

We apply: any set $S \subset E(G_{n*n})$ of size $|S| = \lfloor \frac{n^2}{2} \rfloor$ satisfies $|\partial_e(S)| \geq n$
This result can be seen geometrically: if we take for example our grid $4*4$ in figure 1, no matter how we separate it in two, we will necessarily "cut" 4 connections because cutting it in two in line, which is the minimum, minimizes the number of communications cut while diagonally or any special form that touches both axes we lose more.

Thus, according to our result, no matter how we arrange half of our departments in the lower part of the tower and no matter how we arrange the other half in the upper part of the tower, there will necessarily exist n departments in one of the two parts that communicate with the other
From which: $T_{min} \geq n$

□

e) The number of departments is a cube $N = n^3$, the departments are labeled with triplets (i, j, k) where $1 \leq i, j, k \leq n$ and department (i, j, k) communicates with departments $(i + 1, j, k)$ (if $i + 1 \leq n$), $(i, j + 1, k)$ (if $j + 1 \leq n$) and $(i, j, k + 1)$ (if $k + 1 \leq n$).

Answer 2.5. $T_{min} = n^2$

Proof:

In this case, the set of departments and the communication plan are written in the form:
– $E = \{D_{i,j,k} / 1 \leq i, j \leq n\}$
– $D = P_x \cup P_y \cup P_z$
With $P_x = \{C_{D_{i,j,k}, D_{i+1,j,k}} \ / \ i + 1 \leq n\}$
$P_y = \{C_{D_{i,j,k}, D_{i,j+1,k}} \ / \ j + 1 \leq n\}$
$P_z = \{C_{D_{i,j,k}, D_{i,j,k+1}} \ / \ k + 1 \leq n\}$

Let's show that $T_{min} \leq n^2$:
We consider the following distribution:
$(\forall D_{i,j,k} \in E), h(D_{i,j,k}) = (i-1)n^2 \ + \ (j-1)n \ + \ k$
$(\forall C_{D_{i,j,k},D_{i+1,j,k}} \in P_x); f(C_{D_{i,j,k},D_{i+1,j,k}}) = |h(D_{i,j,k}) - h(D_{i+1,j,k})|$
$\implies (\forall C_{D_{i,j,k},D_{i+1,j,k}} \in P_x); f(C_{D_{i,j,k},D_{i+1,j,k}}) = |(i-1)n^2 + (j-1)n + k - ((i+1-1)n^2 + (j-1)n \ + \ k)|$
$\implies (\forall C_{D_{i,j},D_{i+1,j}} \in P_x); f(C_{D_{i,j},D_{i+1,j}}) = | - n^2|$
$\implies (\forall C_{D_{i,j},D_{i+1,j}} \in P_x); f(C_{D_{i,j},D_{i+1,j}}) = n^2$

$(\forall C_{D_{i,j,k},D_{i,j+1,k}} \in P_y); f(C_{D_{i,j,k},D_{i,j+1,k}}) = |h(D_{i,j,k}) - h(D_{i,j+1,k})|$
$\implies (\forall C_{D_{i,j,k},D_{i,j+1,k}} \in P_y); f(C_{D_{i,j,k},D_{i,j+1,k}}) = |(i-1)n^2 \ + \ (j-1)n \ + \ k \ - \ ((i-1)n^2 \ + \ (j+1-1)n \ + \ k)|$
$\implies (\forall C_{D_{i,j,k},D_{i,j+1,k}} \in P_y); f(C_{D_{i,j,k},D_{i,j+1,k}}) = | - n|$
$\implies (\forall C_{D_{i,j,k},D_{i,j+1,k}} \in P_y); f(C_{D_{i,j,k},D_{i,j+1,k}}) = n$

$(\forall C_{D_{i,j,k},D_{i,j,k+1}} \in P_z); f(C_{D_{i,j,k},D_{i,j,k+1}}) = |h(D_{i,j,k}) - h(D_{i,j,k+1})|$
$\implies (\forall C_{D_{i,j,k},D_{i,j,k+1}} \in P_z); f(C_{D_{i,j,k},D_{i,j,k+1}}) = |(i-1)n^2 \ + \ (j-1)n \ + \ k \ - \ ((i-1)n^2 \ + \ (j-1)n \ + \ (k+1))|$
$\implies (\forall C_{D_{i,j,k},D_{i,j,k+1}} \in P_z); f(C_{D_{i,j,k},D_{i,j,k+1}}) = | - 1|$
$\implies (\forall C_{D_{i,j,k},D_{i,j,k+1}} \in P_z); f(C_{D_{i,j,k},D_{i,j,k+1}}) = 1$

From the results obtained: $(\forall C_{D_{i,j,k}}, C_{D_{i',j',k'}} \in P); f(C_{D_{i,j,k}}, C_{D_{i',j',k'}}) \leq n^2$
$\implies T_{transit} = n^2$
From which, whatever the distribution : $T_{min} \leq n^2$

Let's show that $T_{\min} \geq n^2$
Let's take up the communication plan in an illustrated way: a cube

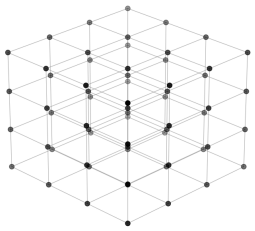

Figure 2: Communication plan for $n = 4$

For the continuation of our proof, we apply the same idea as for the square grid, but in three-dimensional space. We need the following result, which is its generalization:
– If we separate our cube into two equal parts (or almost equal if N odd), we will necessarily "break" at least n^2 communications. That is to say that the first group shares at least with the second group n^2 communications.
This result is obtained by applying the same corollary 4 2.1 [1] as for the square to our situation. The result is expressed thus: any set $S \subset E(G_{n*n*n})$ of size $|S| = \lfloor \frac{n^3}{2} \rfloor$ satisfies $|\partial_e(S)| \geq n^2$.
We can see this result geometrically: if we take for example a cube $4 * 4 * 4$ 2, no matter how we separate it in two, we will necessarily "cut" at least 16 connections.

Thus, according to this result, no matter how we arrange half of our departments in the lower part of the tower and no matter how we arrange the other half in the upper part of the tower, there will necessarily exist at least n^2 pairs of communicating departments distributed on either side of this separation.
From which: $T_{\min} \geq n^2$

□

2. Describe the plans whose optimal transit time is:

In order to treat this question, it is important to recall a notion: there exist several communication plans for a given $T_{\min}$. To answer this question, we will base ourselves on the following two rules.

– The communication plans that verify an optimal transit time must be included in maximal com. plan $P_{\max}$ (we will detail $P_{\max}$ later).
– These communication plans must contain a sub-communication plan P_{min} (partial graph) that we can consider as the sufficient and necessary condition to obtain this T_{min}
Remark: there can be more than one plan P_{min} The treatment of this question will consist of two steps:
– Find the maximal plan
– Find the minimal plan(s)
All this is done obviously for a given n
Proof of these two rules:
$\begin{cases} P \subseteq P_{max} \implies T_{min}(P) \leq T_{min}(P_{max}) \\ P \supseteq P_{min} \implies T_{min}(P) \geq T_{min}(P_{max}) \end{cases}$
(By increasing the number of communications, the transit time can either increase or remain equal)
Thus: $P_{min} \subseteq P \subseteq P_{max} \implies P$ has for optimal transit time T_{min}

Conjecture: $T_{min} = k$
$\implies P_{max} = \{C_{D_i,D_{i+1}}, C_{D_i,D_{i+2}}, \dots C_{D_i,D_{i+k}}, C_{D_i,D_{i-1}}, C_{D_i,D_{i-2}}, \dots, C_{D_i,D_{i-k}}\}$
With: $k+1 \leq i \leq N-k-1$
We will assume this conjecture to answer the questions. This result is intuitive but we have still not succeeded in proving it.

a) 1

Answer 2.6. $P_{min} = \{C_{D_a,D_b}\}$
$P_{max} = \{C_{D_1,D_2}, C_{D_2,D_3}, \dots, C_{D_i,D_{i+1}}, \dots, C_{D_{N-1},D_N} \ / \ 1 \leq i \leq N-1\}$

Proof:
$P_{max} = \{C_{D_1,D_2}, C_{D_2,D_3}, \dots, C_{D_i,D_{i+1}}, \dots, C_{D_{N-1},D_N} \ / \ 1 \leq i \leq N-1\}$
$P_{min} = \{C_{D_a,D_b}\}$
Figures 3 and 4 represent these plans (paths):

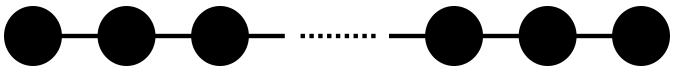

Figure 3: Maximal plan for $T_{min} = 1$

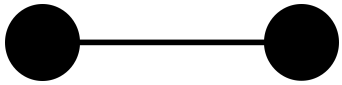

Figure 4: Minimal plan for $T_{min} = 1$

Let's suppose that: $(\exists! C_{D_a,D_b})$ / $f(C_{D_a,D_b}) = |h(D_a) - h(D_b)|$
We take $h(D_a) = k$ and $h(D_b) = k \pm 1$
$\implies f(C_{D_a,D_b}) = |\pm 1| = 1$
From which: the expression of P_{min}
The expression of P_{max} is obtained from the conjecture but is also obvious: it suffices to place the departments in an ordered manner to obtain $T_{min} = 1$ and we cannot do better than 1
If we add at least one communication, we fall on the case of the following question.

□

b) 2

Answer 2.7. $P_{min} = \{C_{D_a,D_b}, C_{D_b,D_c}, C_{D_b,D_d}\}$
or $P_{min} = \{C_{D_1,D_2}, C_{D_2,D_3}, \dots C_{D_{k-1},D_k}, C_{D_1,D_k}\}$ with $3 \leq k \leq n$ (Cycle C_k)
$P_{max} = \{C_{D_i,D_{i'}}, C_{D_i,D_{i+1}}, C_{D_{i'},D_{i'+1}}$ / $1 \leq i \leq \lceil \frac{N}{2} \rceil$ *and* $1 \leq i \leq \lfloor \frac{N}{2} \rfloor\}$

Proof:
$P_{max} = \{C_{D_i,D_{i'}}, C_{D_i,D_{i+1}}, C_{D_{i'},D_{i'+1}}$ / $1 \leq i \leq \lceil \frac{N}{2} \rceil$ *and* $1 \leq i \leq \lfloor \frac{N}{2} \rfloor\}$
$P_{min} = \{C_{D_a,D_b}, C_{D_b,D_c}, C_{D_b,D_d}\}$ or
$P_{min} = C_n$ (cycle) Figures 5 and 6 represent these plans:

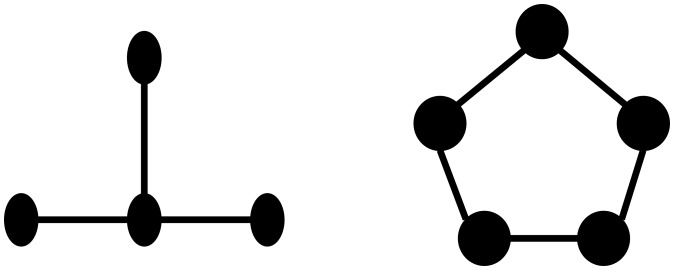

Figure 5: Plan minimal pour $T_{min} = 2$

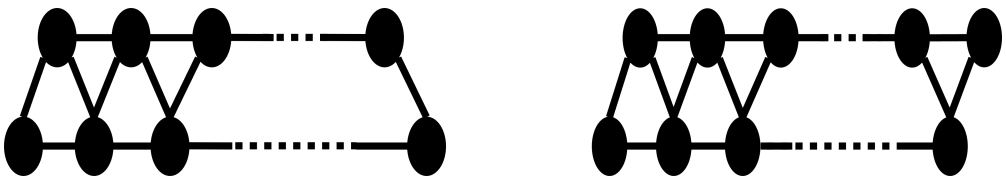

Figure 6: Plan maximal pour $T_{min} = 2$

If the plan does not contain one of the two minimal plans then the plan becomes a chain and therefore of minimal transit time $T_{min} = 1$ (see previous case), if it contains one of the two, we have proved one of the two previously in question **1.**b) and the other case is trivial (we can even do it exhaustively).
The expression of P_{max} is obtained from the conjecture

□

c) $N - 2$

Answer 2.8. $P_{min} = \{C_{D_i,D_j} \ / \ D_i, D_j \in E \setminus \{D_\alpha\}\}$ with D_α a fixed department
$P_{max} = \{C_{D_i,D_j} \ / \ D_i, D_j \in E\} \setminus \{C_{D_\alpha,D_\beta}\}$ with D_α and D_β two fixed departments

Proof:
$P_{min} = \{C_{D_i,D_j} \ / \ D_i, D_j \in E \setminus \{D_\alpha\}\}$ with D_α a fixed department
$P_{max} = \{C_{D_i,D_j} \ / \ D_i, D_j \in E\} \setminus \{C_{D_\alpha,D_\beta}\}$ with D_α and D_β two fixed departments

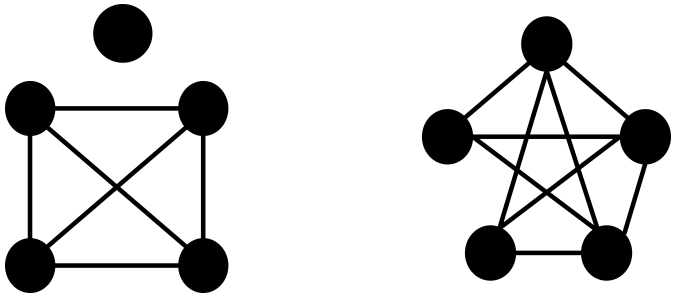

Figure 7: Plans minimal et maximal de $T_{min} = N - 2$ pour $N = 5$

– For P_{min}, as we will see in question d), a complete graph K_N has for transit time $T_{min} = N-1$ and it is the only one having this transit time i.e. if we decrease the number of communications the transit time will necessarily decrease (see d)). Thus, for a complete graph of order $N - 1$, $T_{min} = N - 2$. The last isolated department does not influence the transit time.
– For P_{max}, if we add the only remaining communication, we obtain $T_{min} = N - 1$ as affirmed by the result of the following question. Therefore the maximal plan is indeed the one defined previously.

□

d) $N - 1$

Answer 2.9. $P = P_{min} = P_{max} = \{C_{D_i,D_j} \ / \ D_i, D_j \in E\}$ (complete graph)

Proof:
$P = P_{min} = P_{max} = \{C_{D_i,D_j} \ / \ D_i, D_j \in E\}$
The following figure represents the plan:

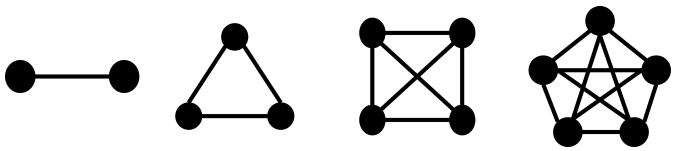

Figure 8: Graphes complets de $N = 2$ jusqu'à $N = 5$

Whatever the chosen arrangement, it is necessary that:
$(\forall C_{D_i,D_j} \in P); f(C_{D_i,D_j}) = N - 1 \; if \; h(D_i) = 1 \; and \; h(D_j) = N$
$\Longrightarrow$ it is necessary that each element communicates with all the others.
Conclusion: $P = P_{min} = P_{max} = \{C_{D_i,D_j} \;\; / \;\; D_i, D_j \in E\}$

□

3. For each of the following constraints, bound as a function of N and the other variables used the worst optimal transit time (therefore the largest) among the set of communication plans that verify this constraint:

a) There are at most k pairs of departments that communicate.

Answer 2.10. $\begin{cases} T_{min} = \lceil \frac{k}{2} \rceil & \text{if } \; k \leq N-1 \\ \lceil \frac{N-1}{2} \rceil < T_{min} < N-1 & \text{if } \; N-1 < k < \frac{n(n-1)}{2} \\ T_{min} = N-1 & \text{if } \; k = \frac{n(n-1)}{2} \end{cases}$

Proof:

The first and the last case are quite simple:
– The first case is analogous to questions 1 a) and 3 c). The objective is again to search for the star graph (as in 3.c)) to have the worst optimal transit time and it is calculated by analogy to question 1.a
– The third case is obvious but is also analogous to 2 d). The given number of communications can only form a complete graph which has for optimal transit time $T_{min} = N - 1$

For the second case, we have provided a rather wide bound because the situation involves too many possibilities that could be done really case by case. But we still have some research leads that we did not have time to explore in more detail: explore the case of complete bipartite graphs or any graph that centralizes communications at a minimum of points. □

b) Each department communicates with at most k other departments

Answer 2.11. $\begin{cases} T_{\min} = 1 & \text{if } \; k = 1 \\ T_{\min} = 2 & \text{if } \; k = 2 \text{ and } N \geq 4 \\ \lceil \frac{k}{2} \rceil \leq T_{\min} \leq N - \lceil \frac{N}{k+1} \rceil & \text{if } \; 3 \leq k \leq N-2 \\ T_{\min} = N-1 & \text{if } \; k = N-1 \end{cases}$

Proof:

The graph considered this time is the regular graph
- **Case** $k = 1$: Each department communicates with exactly 1 other. The graph is a union of disjoint edges (matching). We can place each pair on consecutive floors, therefore $T_{\min} = 1$.
- **Case** $k = 2$: Each department has degree 2. The graph is a union of cycles and paths. A cycle C_N gives $T_{\min} = 2$ (question 1.b) and it is the worst possible.
- **Case** $k = N - 1$: Each department communicates with all the others. The graph is complete K_N, therefore $T_{\min} = N - 1$ (question 2.d).
- **Case** $3 \leq k \leq N - 2$:

- **Lower bound**: If we consider an isolated point, it will have the star graph case of the previous question a). If it were this single department that communicated we would say that $T_{min} = \lceil \frac{k}{2} \rceil$. Now since there are more, we can say that $T_{min} \geq \lceil \frac{k}{2} \rceil$
- **Upper bound**: We divide the N floors into $k + 1$ blocks of size $b = \lceil N/(k + 1) \rceil$. Each vertex has at most k neighbors, therefore it cannot have a neighbor in all blocks. Thus, at least one complete block contains none of its neighbors. This forces the maximum distance between a vertex and its neighbors to be at most $N - b = N - \lceil N/(k + 1) \rceil$.

The bound remains wide because the value depends on the structure of the regular graph which is too vague □

c) There are exactly $N - 1$ pairs of departments that communicate and all departments are connected, in the sense that for two arbitrary departments X and Y, we can find departments $D_1, \ldots, D_l$ such that D_i communicates with D_{i+1} for all i.

Answer 2.12. $T_{min} = \left\lfloor \frac{N}{2} \right\rfloor$

Proof:

– The second condition: "for two arbitrary departments X and Y, we can find departments $D_1, \ldots, D_l$ such that D_i communicates with D_{i+1} for all i" expresses quite simply in graph theory that the graph is connected (there always exists a path between two vertices of the graph and in our case the departments).
– When we superimpose the first condition ("There are exactly $N - 1$ pairs of departments that communicate") on the second condition, it forces the absence of cycle because if there is at least one cycle and $N - 1$ communications a department will necessarily remain isolated.
Thus, the graph obtained is a tree.
To obtain a bad optimal transit time, it is quite simply necessary to centralize the communications towards a unique department.
The graph obtained is a star graph
However, we have already studied the optimal transit time of this arrangement (star graph) in a), and we found that: $T_{min} = \left\lfloor \frac{N}{2} \right\rfloor$

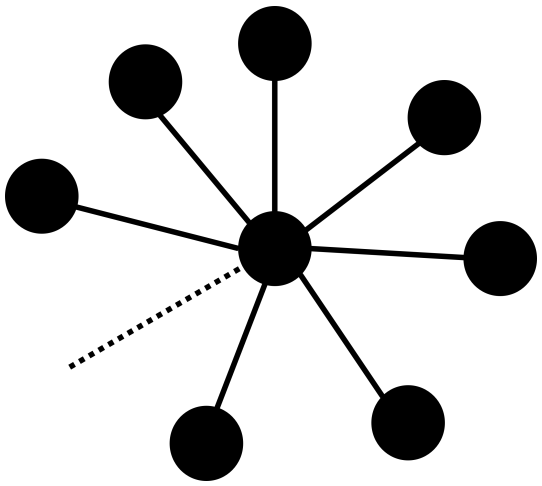

Figure 9: Graphe étoile

□

3 Part 3 - Optimal Effort Time

Let us take up questions 1., 2., 3. in this framework, by replacing the transit time by the effort time.

Before treating this question, let us contextualize the effort time:
Concerned about the physical health of his employees, the CEO of GraphEtCo wants them to take the stairs as much as possible. He asks on the contrary Younes to make sure that the departments that communicate are as far apart as possible. Given a communication plan, the effort time of a distribution is the smallest difference of floors among the pairs of departments that communicate. The optimal effort time of the plan is the largest effort time that Younes can obtain by distributing the departments.

1. Calculate or bound the optimal effort time as a function of N for each of the following communication plans:

1. Calculate or bound the optimal effort time as a function of N for each of the following plans:

a) One of the departments communicates with each other department.

Answer 3.1. $T_{max} = 1$

Proof:

$(\forall h(D_i) \in [\![1, N]\!])(\exists D_{i'} \in E \setminus \{D_i\}) \;\; / \;\; h(D_{i'}) = h(D_i) \pm 1$
$\implies (\forall h(D_i) \in [\![1, N]\!])(\exists D_{i'} \in E \setminus \{D_i\}) \;\; / \;\; f(C_{D_i,D_{i'}}) = |h(D_i) - h(D_i) \pm 1|$
$\implies (\forall h(D_i) \in [\![1, N]\!])(\exists D_{i'} \in E \setminus \{D_i\}) \;\; / \;\; f(C_{D_i,D_{i'}}) = |\pm 1|$
$\implies (\forall h(D_i) \in [\![1, N]\!])(\exists D_{i'} \in E \setminus \{D_i\}) \;\; / \;\; f(C_{D_i,D_{i'}}) = 1$
$\implies$ Whatever the distribution: $(\exists C_{D_i,D_{i'}}) \;\; / \;\; f(C_{D_i,D_{i'}}) = 1$
$\implies$ Whatever the distribution: $T_{eff} = 1$

$\implies T_{max} = 1$

□

b) By numbering the departments from 1 to N, every department i communicates with $i+1$ and N communicates with 1.

Answer 3.2. $T_{\max} = \left\lfloor \frac{N-1}{2} \right\rfloor$ for $N \geq 3$, and $T_{\max} = 1$ for $N = 2$

Proof:

Let us show that $T_{\max} \leq \left\lfloor \frac{N-1}{2} \right\rfloor$:

Let us take an optimal distribution. Let us denote $m = T_{\max}$. Then for every edge $C_{D_i,D_{i+1}}$, we have $f(C_{D_i,D_{i+1}}) \geq m$.

In a cycle, if we denote the floors in the order in which the vertices of the cycle appear, the sum of the f on the N edges is at most:

$$1 + 2 + \cdots + (N-1) = \frac{N(N-1)}{2}$$

Because the largest possible difference between two floors is $N-1$.

Therefore $N \cdot m \leq \frac{N(N-1)}{2}$, that is $m \leq \frac{N-1}{2}$. Since m is integer, $m \leq \left\lfloor \frac{N-1}{2} \right\rfloor$.

Let us show that we can reach this value:

For N odd: $N = 2p+1$. We place the vertices in the order: $1, N, 2, N-1, 3, \ldots, p, p+2, p+1$. The differences between neighbors in the cycle are: $p, p, p-1, p-1, \ldots, 1, 1$. The minimum is 1, but we can adjust so that the minimum is $p = \left\lfloor \frac{N-1}{2} \right\rfloor$.

For N even: $N = 2p$. We place: $1, N, 2, N-1, 3, \ldots, p, p+1$. The differences are: $N-1, N-3, \ldots, 1, 1, 3, \ldots, N-3$. The minimum is 1, but with a better arrangement we can obtain $p - 1 = \left\lfloor \frac{N-1}{2} \right\rfloor$.

For $N = 2$: a single edge, $T_{\max} = 1$. □

c) The departments are formed of two groups of size k and l (therefore N=k+l) and each department of one group communicates with all the departments of the other group, and does not communicate with any department of its own group.

Answer 3.3. $T_{max} = 1$

Proof:

Let us show by absurdity that $T_{\max} = 1$:

Let us suppose that $T_{\max} \neq 1$.

There necessarily exists a D_a such that $h(D_a) = 1$.

We will work only on the case where $D_a \in E_k$ (because the other is worked exactly in the same manner, we recall that the roles of k and l are interchangeable).

$D_a \in E_k$ and $T_{\max} \neq 1 \implies D_b \in E_k$ with D_b the department $h(D_b) = 2$ (because $h(D_b) - h(D_a) = 1$)

$D_b \in E_k$ and $T_{\max} \neq 1 \implies D_c \in E_k$ with D_c the department $h(D_c) = 3$

And so on until the department $D_\alpha \in E_k$ such that $h(D_\alpha) = k$

Since $|E_k| = k$, there exists the department $D_\beta \in E_l$ such that $h(D_\beta) = k + 1$.

$$\begin{aligned} D_\alpha \in E_k \text{ and } D_\beta \in E_l &\implies \exists C_{D_\alpha, D_\beta} \in P \text{ such that } h(C_{D_a, D_p}) = |h(D_\alpha) - h(D_\beta)| \\ &\implies h(C_{D_\alpha, D_\beta}) = |k - (k+1)| \\ &\implies h(C_{D_\alpha, D_\beta}) = |-1| \\ &\implies h(C_{D_\alpha, D_\beta}) = 1 \quad \text{(which is absurd)} \end{aligned}$$

Conclusion: $T_{\max} = 1$**.**

□

d) The number of departments is a square $N = n^2$, with the communications in grid.

Answer 3.4. $T_{\max} = 1$

Proof:

The graph is a grid $n \times n$.
The total number of pairs of consecutive floors is $N - 1 = n^2 - 1$
The total number of communications is $2n(n-1)$
Thus: the total number of communications is greater than the number of pairs of consecutive floors.
By application of the pigeonhole principle, there will necessarily be a communication that links two consecutive floors
Hence $T_{max} = 1$ □

e) The number of departments is a cube $N = n^3$, with the communications in cube.

Answer 3.5. $T_{\text{eff,opt}} = 1$ for $n \geq 2$

Proof:

Same reasoning as for the grid, but with even more edges: $3n^2(n-1)$.
We have $3n^2(n-1) > n^3 - 1$ for $n \geq 2$ (easy verification).
Therefore by the pigeonhole principle, there exists an edge linking two consecutive floors.
Thus $T_{\max} = 1$. □

2. Describe the plans whose optimal effort time is:

a) 1

Answer 3.6. A connected graph. In particular:
- Any graph containing a Hamiltonian path
- Any complete graph K_N
- Any complete bipartite graph $K_{k,l}$

- Any tree
- Any grid, any cube

Proof:

For a graph to have $T_{\max} = 1$, it suffices that in every distribution, there exist two adjacent vertices at consecutive floors. This is the case for almost all non-trivial connected graphs.
A counterexample would be a graph that allows a distribution where all edges have $f \geq 2$, like a cycle C_N for $N \geq 5$. □

b) $N - 1$

Answer 3.7. A single communication

Proof:

$T_{\text{eff,opt}} = N - 1$ would mean that we can have all edges with $f \geq N - 1$.
But $f \geq N - 1$ implies that the edges link only floor 1 and floor N. □

3. For each of the following constraints, bound the worst optimal effort time:

a) There are at most k pairs that communicate.

Answer 3.8. $t_{\max}(k) = N - (2k - 1)$ if $k \leq \lfloor N/2 \rfloor$

Proof:

If $k \leq \lfloor N/2 \rfloor$: we take k disjoint edges. To maximize T_{eff}, we place the pairs at the extremities:
- Pair 1: floors 1 and N, difference $N - 1$
- Pair 2: floors 2 and $N - 1$, difference $N - 3$
- ...
- Pair k: floors k and $N - k + 1$, difference $N - (2k - 1)$

The minimum is $N - (2k - 1)$. This is the best possible.
If $k \leq \lfloor N/2 \rfloor$: we have not yet found a precise result. But it is certain that it will quickly converge toward 1. □

b) Each department communicates with at most k others.

Answer 3.9.

$$t_{\max}(k) \leq \left\lfloor \frac{N}{k+1} \right\rfloor$$

Proof:

For a graph of max degree $\leq k$, let us take an optimal distribution with $T_{\max} = m$.
Each vertex has its neighbors at distance $\geq m$. Therefore if we cut the floors into blocks of size $m + 1$, each vertex has at most one neighbor per block.
There are $\lceil N/(m+1) \rceil$ blocks, therefore $k \leq \lceil N/(m+1) \rceil$.

Thus $m+1 \leq \lceil N/k \rceil$, that is $m \leq \lfloor N/k \rfloor - 1$ approximately. A more precise calculation gives $m \leq \lfloor N/(k+1) \rfloor$.

□

c) There are exactly $N-1$ pairs of departments that communicate and all departments are linked, in the sense that for any two departments X and Y, we can find departments $D_1, \ldots, D_l$ such that D_i communicates with D_{i+1} for all i.

Answer 3.10.

$$T_{\max} = 1$$

Proof:

– The condition "all departments are linked" means that the graph is **connected** in graph theory.

– A graph with N vertices and $N-1$ edges that is connected is **a tree**.

Let us show that every tree has $T_{\max} = 1$:

Let us take any tree and any distribution h of departments on the N floors.

Let us consider the two departments placed at floors 1 and 2: - If these two departments communicate, then we have found an edge with $f = 1$. - If they do not communicate, let us consider the **unique path** in the tree that links them:

$$D_{a_1}(= \text{floor } 1) - D_{a_2} - \cdots - D_{a_m}(= \text{floor } 2)$$

Along this path, the floors must pass from 1 to 2.

Therefore there exists an index i such that:

$$|h(D_{a_i}) - h(D_{a_{i+1}})| = 1$$

Because we pass from floor 1 to floor 2 in a discrete way.

Thus, in **every** distribution, there exists an edge of the tree with $f = 1$.

Therefore $T_{\text{eff,opt}} \leq 1$. Since we cannot have $T_{\max} = 0$ (distinct floors), we have $T_{\max} = 1$ for every tree.

Since our constraint forces the graph to be a tree, the worst optimal effort time among all these graphs is $T_{max} = 1$. □

References

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